Parametric Robust Control by Quantitative Feedback Theory

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The problem of performance robustness, especially in the face of significant parametric uncertainty, has been increasingly recognized as a predominant issue of engineering significance in many design applications. Quantitative feedback theory is very effective for dealing with this class of problems even when there exist hard constraints on closed-loop response. In this paper, single-input/single-output quantitative feedback theory is viewed formally as a sensitivity constrained multiobjective optimization problem whose solution cannot be obtained analytically but (when feasible) can be obtained graphically. In contrast to the more recent robust control methods where phase uncertainty information is often neglected, the direct use of parametric uncertainty and phase information in quantitative feedback theory results in a significant reduction in the cost of feedback. An example involving a standard quantitative feedback theory problem is included for completeness.

Nomenclature

$\|A\|_\infty$	$= \sup\{\|A(i\omega)\| : \omega \in R\},\ A \in H^\infty$
$A := B$	$= A$ is defined by B
$A \simeq B$	$= A$ is approximately equal to B
e,r	= relative degree of transfer function
G_H	$= H^\infty$ controller
G_Q	= quantitative feedback theory controller
H^∞	= Banach space of bounded analytic functions
L_∞	= Banach space of essentially bounded Baire functions
RH^∞	= Banach space of bounded analytic functions with elements from the ring of stable, proper real rational functions (unit of RH^∞ = an element of RH^∞ whose inverse $\in RH^\infty$)
S_H	$= H^\infty$ sensitivity function
S_Q	= quantitative feedback theory sensitivity function
$\|\cdot\|_\infty$	= norm on L_∞
Ω	= compact parameter space with elements α
ω,λ	= radian frequency

I. Introduction

THE last decade has witnessed a steady and growing research effort in robust control; see, for example, Ref. 1. The majority of this effort has been devoted to systems that are assumed to have unstructured uncertainty. This allows such problems to be transformed into a form where the small gain theorem[2] and powerful recent mathematical techniques from functional analysis and operator theory[3,4] can be successfully employed for system analysis and synthesis. However, many problems of practical interest appear as models with both large parametric uncertainty and high-frequency nonparametric uncertainty. Typical examples include flight control and turbomachinery control over a flight envelope parametrized by power level, height, and Mach number, as well as general automotive engine control problems.

All of these problems yield a collection of linear models obtained by a linearization of the parametrically dependent nonlinear differential equation set about a finite number of different operating points. This problem class is often endowed with hard stability and performance constraints such as on rise time and overshoot. This problem class also requires plant uncertainty to be expressed as variations in both gain and phase. Traditional control of this problem has relied on gain scheduling or the on-line switching of controllers designed for models obtained at the different operating points, as and when due. The design of the switching logic and some resulting stability problems are nontrivial. When feasible, parametric robust control makes gain scheduling redundant. A nonparametric description of uncertainty is of course possible but will almost always result in loss of phase information. This often leads to higher bandwidth controllers.

The quantitative feedback theory (QFT) robust control methodology introduced by Horowitz and co-workers[5-10] is perhaps the only known technique that considers both large parametric uncertainty and phase information simultaneously. The major payoff is the ability to satisfy both robust stability and multiobjective hard performance constraints with the minimum possible cost of feedback. The downside is that the method, though systematic and powerful in the hands of an experienced control engineer, has not, until recently, lent itself easily to formal mathematization as in the more recent paradigms such as H^∞ control and μ synthesis. The present effort is an attempt to bridge the gap. A key aim of the present work is to systematize the QFT design process. The QFT problem can be posed as a formal sensitivity constrained optimization problem[11] that reduces to the problem statements in H^∞ control when the hard performance constraints and parametric uncertainty descriptions are relaxed. Consequently, the newer methods may be viewed as restricted quantitative feedback design methods.[12]

This paper is divided into six sections. In Sec. II, we develop a complete description of the representation of plant uncertainty. In Sec. III, we formally state the QFT problem and subsequently convert it into a constrained sensitivity optimization problem whose general analytic solution in unknown. In Sec. IV, we translate these requirements into the classical QFT format and use graphical search techniques and H^∞ design methodology to generate suboptimal solutions to the sensitivity optimization problem. In Sec. V, we give a nontrivial example with the corresponding H^∞ solution. This example clearly demonstrates the more aggressive controller bandwidth prob-

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lem that is called for whenever there is a loss of phase uncertainty information. Finally, in Sec. VI, we give some concluding remarks.

II. Representation of Plant Uncertainty

We assume that the process to be modeled is uncertain due to both parametric and unstructured, nonparametric uncertainty. Parametric uncertainty is uncertainty that can be represented by parameter variations in an appropriately structured process model, whereas nonparametric uncertainty is any process uncertainty that cannot be explained adequately in the context of the structured uncertainty. Typical sources of nonparametric uncertainty are unmodeled dynamics, unmodeled parameter variations, and measurement errors. The process consists of the union of parametric and nonparametric components.

We may therefore represent the uncertain process as follows:

$$\{P(s)\} = P(\alpha,s) + P_n(s) \tag{1}$$

Here, $\{P(s)\}$ is the plant set, $P(\alpha,s)$ the parametric plant subset, and $P_n(s)$ the corresponding nonparametric plant subset in the additive unstructured uncertainty form.[16] The parametric plant set is described by $\{P(\alpha,s),\ \alpha \in \Omega \subset R^m\}$, where α is an m-dimensional parameter vector that ranges over the compact parameter space Ω. Each $\alpha_o \in \Omega$ generates a plant $P(\alpha_o,s) \in P(\alpha,s)$. The set $P(\alpha,s)$ is represented as

$$P(\alpha,s) = \frac{k\prod_i (s+z_i)\prod_k (s^2 + 2\zeta_k\omega_k s + \omega_k^2)}{s^n\prod_{j=1} (s+p_j)\prod_l (s^2 + 2\zeta_l\omega_l s + \omega_l^2)} \tag{2}$$

Each of the parameters k, z_i, ζ_k, ω_k, p_j, ζ_l, ω_l is uncertain, continuously dependent on $\alpha \in \Omega$, and lies in a specific compact interval. However, the maximum order and maximum relative degree of $P(\alpha,s)$ is fixed. This representation is most appropriate for the process behavior in the low to mid frequency range. On the other hand, the unstructured plant subset $P_n(s)$ is assumed to affect the process dynamics significantly only at high frequencies. We may then suppose that the plant set admits the following split frequency representation:

$$\{P(s)\} = \begin{cases} P(\alpha,s), & \forall\omega < \omega_h,\ s = i\omega \\ P_n(s), & \forall\omega \geq \omega_h,\ s = i\omega \end{cases} \tag{3}$$

The frequency ω_h marks the beginning of the high-frequency region. For $\omega \geq \omega_h$, the overall plant set $\{P(s)\}$ degenerates to the high-frequency model $P_n(s)$, which is assumed to satisfy

$$|P_n(i\omega)| \leq \frac{k_h}{\omega^e}, \qquad \forall\omega \geq \omega_h \tag{4}$$

$$\max_{\alpha\in\Omega} |P(\alpha,i\omega_h)| \leq \frac{k_h}{\omega_h^e} \tag{5}$$

where e is the maximum relative degree of $P(\alpha,s)$, and k_h represents the plant high frequency gain defined by

$$k_h := \max_{\alpha\in\Omega}\left\{\lim_{s\to\infty} s^e P(\alpha,s)\right\} \tag{6}$$

Note in particular that both $P(\alpha,s)$ and $P_n(s)$ have the same high-frequency gain.

We also assume the following:

1) $P_n(s)$ is stable with its magnitude gain bounded by the high-frequency gain as just given, but otherwise is totally without structure.

2) On the other hand, the structured model $P(\alpha,s)$ may be unstable from some or all $\alpha \in \Omega$, but must have a stable inverse for all $\alpha \in \Omega$; i.e., $P(\alpha,s)$ is minimum phase for all $\alpha \in \Omega$.

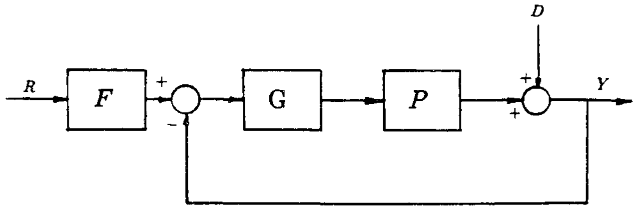

Fig. 1 Standard two-degree-of-freedom feedback structure.

3) The plant set $\{P(s)\}$ is topologically path connected.[13,14]

The transition frequency ω_h is called the Horowitz frequency.[13] It is indeed only from this frequency that the plant set transitions completely to the unstructured model $P_n(s)$ and is assumed to have arbitrary phase uncertainty. To characterize the low-frequency uncertainty ($\omega \leq \omega_h$), the mapping $P(\alpha,i\omega)$: $\Omega \to R(g,\phi,\omega)$ generates a compact set $R(g,\phi,\omega)$ in the gain-phase plane at each frequency, commonly called the plant template. This essentially captures the gain g and phase ϕ variations of the plant at the given frequency ω. System design using the full description of $R(g,\phi,\omega)$ will be referred to hereafter as classical or traditional QFT and has been under continuous development for over three decades.[5]

In the present description of plant uncertainty, it is perfectly legal for the poles of $P(\alpha,s)$ to migrate to and fro across the imaginary axis, or indeed for all of the poles to lie entirely in the open right-half complex plane as α ranges over Ω. However, all of the zeros of $P(\alpha,s)$ are constrained to lie in the interior of the closed left-half complex plane. This extremely realistic description of plant uncertainty was originally proposed by Aström et al.,[15] used by Nwokah,[13] and has recently been implicitly used by Bailey and Hui[16] and Bailey and Cockburn.[17] On the other hand, the unstructured uncertainty description mandated by the H^∞ methodology dictates that there be no crisscrossing of poles across the imaginary axis, or equivalently that every member of the plant set $\{P(s)\}$ must have exactly the same number of unstable poles.[18] For our sensitivity based QFT design to work, we are forced to assume the same model structure as in H^∞ control. However, we retain the full sensitivity uncertainty description as in standard QFT formulations.[6]

III. Quantitative Feedback Theory Problem Specification

The QFT problem can now be stated as follows. There is given an uncertain family of linear time-invariant finite-dimensional plants $\{P(s)\}$ as described in Sec. II. There is also given an ideal target closed-loop transmission function $T_o(s)$ and an ideal disturbance response transfer function $T_D^o(s)$. The QFT problem is to find (if possible) an admissible pair of strictly proper, rational, and stable functions $\{G(s), F(s)\}$ in the two-degree-of-freedom feedback arrangement shown in Fig. 1, such that the following conditions are satisfied with some measure of optimality:

Robust stability:

$$T(\alpha,s) \quad \text{is stable} \quad \forall\alpha \in \Omega \tag{7}$$

Robust performance:

$$\max_{\alpha\in\Omega} |T(\alpha,s) - T_o(s)| \leq \delta_T(s),\ \forall s \tag{8}$$

Disturbance attenuation

$$\max_{\alpha\in\Omega} |T_D(s)| \leq |T_D^o(s)| \leq \delta_D(s),\ \forall s \tag{9}$$

where $\delta_T(s) \geq 0$ and $\delta_D(s) \geq 0$ are specified a priori, and

$$T(\alpha,s) = H(\alpha,s)\cdot F(s), \qquad T_o(s) = H_o(s)F(s) \tag{10}$$

$$H(\alpha,s)=\frac{L(\alpha,s)}{1+L(\alpha,s)},\qquad H_o(s)=\frac{L_o(s)}{1+L_o(s)} \tag{11}$$

$$L(\alpha,s)=P(\alpha,s)\cdot G(s),\qquad L_o(s)=P_o(s)\cdot G(s) \tag{12}$$

under the constraint that $G(s)$ is an internally stabilizing controller[25] for the plant set $\{P(s)\}$. $P_o(s)$, unlike the traditional QFT methodology, is the nominal plant model that cannot be chosen arbitrarily. The robust performance specification given in Eq. (8) is slightly different from the one originally defined by Horowitz and Sidi[6] but is in an analytic form,[19] which enables direct time domain to frequency domain conversions via Parseval's theorem.[20] Krishnan and Cruickshanks[21] have argued that there is insignificant practical difference between the traditional measure and the one used here. Our design example to follow also confirms this viewpoint. We next convert the system design data into equivalent sensitivity constraints.

From Bode's sensitivity equation, suitably normalized for large uncertainty,[22,23] we can write

$$\frac{T(\alpha,s)-T_o(s)}{T_o(s)}=S(\alpha,s)\cdot\frac{P(\alpha,s)-P_o(s)}{P_o(s)} \tag{13}$$

where

$$S(\alpha,s):=\frac{1}{1+L(\alpha,s)}$$

Therefore,

$$\max_{\alpha\in\Omega}|T(\alpha,s)-T_o(s)|=\max_{\alpha\in\Omega}\left|T_o(s)S(\alpha,s)\left[\frac{P(\alpha,s)-P_o(s)}{P_o(s)}\right]\right| \tag{14}$$

Define the non-negative function $\delta_G(s)$ by

$$\max_{\alpha\in\Omega}\left|\frac{P(\alpha,s)-P_o(s)}{P_o(s)}\right|=\delta_G(s),\quad \forall s \tag{15}$$

Now put $s=i\omega$. Then from Eqs. (17) and (15), the tracking constraint (8) reduces to

$$\max_{\alpha\in\Omega}|S(\alpha,i\omega)|\le\frac{\delta_T(\omega)}{\delta_G(\omega)|T_o(i\omega)|}\,\forall\omega\ge 0 \tag{16}$$

Notice that $S(\alpha,i\omega)$ carries both gain and phase uncertainty information with it, but the phase uncertainty information embodied in the variations of $P(\alpha,i\omega)$ is lost when we take the bound $\delta_G(\omega)$. From Fig. 1, it is clear that the disturbance transfer function $T_D(s)$ is given by

$$T_D(\alpha,s)=\frac{1}{1+L(\alpha,s)}=S(\alpha,s) \tag{17}$$

Thus, the requirement (9) reduces to

$$\max_{\alpha\in\Omega}|S(\alpha,i\omega)|\le|T_D^0(i\omega)|\,\forall\omega\ge 0 \tag{18}$$

Finally, the specifications (7) and (8) put a constraint M_p on the allowable maximum dynamic magnification of the frequency response of $H(\alpha,i\omega)$. A simple analysis of M circles[24] shows that this M_p constraint on $|H(\alpha,i\omega)|$ translates to a worst case phase stability margin ϕ_m given by

$$\sin\phi_m\ge\frac{1}{M_p} \tag{19}$$

or equivalently,

$$\max_{\alpha\in\Omega}|S(\alpha,i\omega_h)|\le M_p \tag{20}$$

Consequently, the system specifications (7-9) have now been converted to sensitivity inequalities. It then follows that all of the QFT constraints are simultaneously satisfied if there exists an internally stabilizing controller $G(s)$ for the plant set $\{P(s)\}$, such that $S(\alpha,s)$ is stable $\forall\alpha\in\Omega$ and satisfies, for $s=i\omega$,

$$\max_{\alpha\in\Omega}|S(\alpha,i\omega)|\le\min\left\{\frac{\delta_T}{\delta_G|T_o|},|T_D^0(i\omega)|\right\}\forall\omega<\omega_h \tag{21}$$

We assume further that $L(\alpha,s)$ is strictly proper with a minimum relative degree $r\ge 2$. Under these conditions,

$$\lim_{\omega\to\infty}|S(\alpha,i\omega)|=1\forall\alpha\in\Omega \tag{22}$$

It is thus clear that the QFT specifications (7-9) are satisfied if an only if

$$S(\alpha,s)\quad\text{is stable}\quad\forall\alpha\in\Omega$$

$$\max_{\alpha\in\Omega}|S(\alpha,i\omega)|\le\min\left\{\frac{\delta_T(\omega)}{\delta_G(\omega)|T_o|},|T_D^0(i\omega)|\right\}\forall\omega\ge 0$$

$$\lim_{\omega\to\infty}|S(\alpha,i\omega)|=1\forall\alpha\in\Omega$$

Bode's sensitivity integral[25] shows that if $S(\alpha,i\omega)$ is stable $\forall\alpha\in\Omega$, and $L(\alpha,i\omega)$ is strictly proper, then

$$\int_0^\infty\max_{\alpha\in\Omega}\left\{\log|S(\alpha,i\omega)|\right\}\,d\omega=\max_{\alpha\in\Omega}\sum_{i=1}^{N_p}\mathrm{Re}\{P_i(\alpha)\} \tag{23}$$

where $P_i(\alpha)$ is an unstable pole of $P(\alpha,s)$. Since in our present formulation the poles of $P(\alpha,s)$ are not allowed to migrate across the imaginary axis, we may choose as our nominal model $P_o(s)\in\{P(s)\}$; the plant with the worst instability, with the associated sensitivity function $S_o(s)$. That is to say, the nominal model must be chosen such that

$$\sum_{i=1}^{N_p}\mathrm{Re}(P_i^o)\ge\max_{\alpha\in\Omega}\sum_{i=1}^{N_p}\mathrm{Re}P_i(\alpha) \tag{24}$$

where P_i^o $i=1,2,\ldots,N_p$ are the unstable poles of $P_o(s)$, which is the transfer function for the parameter combination that generates the worst instability. Given this condition, it follows from Eq. (23) that

$$\int_0^\infty\log|S_o(i\omega)|\,d\omega\ge\int_0^\infty\max_{\alpha\in\Omega}\log|S(\alpha,i\omega)|\,d\omega \tag{25}$$

Hence, inequality (21) is satisfied if an only if

$$|S_o(i\omega)|\le\min\left\{\frac{\delta_T(\omega)}{\delta_G(\omega)|T_o(i\omega)|},|T_D^0(i\omega)|\right\},\,\forall\omega<\omega_h \tag{26}$$

Let ω_s be the first finite frequency at which $|S_o(i\omega)|=1$. This is called the sensitivity cutoff frequency.[26] The benefits of feedback are only obtained in the interval $0\le\omega<\omega_s$, whereas the cost of feedback is paid for in the frequency interval $\omega_s\le\omega\le\infty$. Let G be the set of all robust stabilizing controllers for $\{P(s)\}$. The QFT optimization problem is then set up as follows:

$$\min_{G\in G}\left[\max_{\alpha\in\Omega}\int_{\omega_s}^\infty\log|S(\alpha,i\omega)|\,d\omega\right] \tag{27}$$

subject to the sensitivity constraints:

$$\max_{\alpha\in\Omega}|S(\alpha,i\omega)|\le\min\left[\frac{\delta_T}{\delta_G|T_o|},|T_D^0|\right],\,\omega<\omega_h \tag{28}$$

$$\lim_{\omega \to \infty} |S(\alpha, i\omega)| = 1 \forall \alpha \in \Omega \tag{29}$$

Define the non-negative and bounded function:

$$M(\omega) = \begin{cases} \min\left[\dfrac{\delta_T}{\delta_G |T_o|}, |T_D^0|\right] \forall \omega < \omega_h \\ M_p \text{ at } \omega = \omega_h, \text{ and } \lim\limits_{\omega \to \infty} M(\omega) = 1 \end{cases} \tag{30}$$

$M(\omega)$ forms a boundary for the magnitude of the optimal sensitivity function. If an optimal solution $S(s)$ to Eq. (27) optimization problem exists, it can always be written as

$$S(s) = A(s) S_{\text{opt}}(s) \tag{31}$$

where $A(s)$ is all pass, S_{opt} and S_{opt}^{-1} are in H^∞, and

$$|S_{\text{opt}}(i\omega)| = M(\omega) \quad \text{almost everywhere} \tag{32}$$

If $G(s)$ is an internally stabilizing controller for the plant set, it turns out that an $S_{\text{opt}}(s) \in H^\infty$, satisfying the constraints (32) on $M(\omega)$, exists if and only if the non-negative function $M(\omega) \in L_\infty$ and, in addition,[23,27]

$$\int_{-\infty}^{\infty} \log M(\omega) \, \mathrm{d}\omega > -\infty \tag{33}$$

In other words, if an optimal sensitivity function $S(s)$ exists, its magnitude $|S(i\omega)|$ lies on the sensitivity boundary $M(\omega)$ almost everywhere. This conclusion was also arrived at by Gera and Horowitz[28] from a different route. However, as pointed out by Gera and Horowitz,[28] a finite order $S_{\text{opt}}(s) \in RH^\infty$ that simultaneously satisfies the bounds on $M(\omega)$ with equality does not exist.

The QFT problem then effectively reduces to the synthesis of a suitable approximation of $S_{\text{opt}}(s)$ with a unit $S_a(s) \in RH^\infty$ such that

$$\|S_{\text{opt}}(s) - S_a(s)\|_\infty < \epsilon \tag{34}$$

for any arbitrarily small $\epsilon > 0$. If so, then a feasible suboptimal sensitivity function $S_o(s)$, which satisfies the inequality constraints on $M(\omega)$, can be written as

$$S_o(s) = A(s) S_a(s) = B_p(s) S_a(s) \tag{35}$$

where $A(s)$ can be suitably approximated as

$$A(s) \simeq B_p(s) \tag{36}$$

and $B_p(s)$ represents the Blaschke product of the unstable poles of $P_o(s)$. A mathematical solution of this optimization and approximation problem does not exist because, in general, $S_{\text{opt}}(s)$ is not continuous on the boundary, i.e., the imaginary axis.[20]

IV. Solution via Automatic Loop Shaping and H^∞ Optimization

Although constructive general existence conditions for the problem posed in Sec. III are unknown, QFT has employed graphical techniques to obtain acceptable solutions for the same problem for a very long time.[6] From Eq. (16), write

$$S(\alpha, i\omega) = \frac{1}{1 + L(\alpha, i\omega)} = \frac{1}{1 + L_o \cdot \dfrac{P(\alpha, i\omega)}{P_o(i\omega)}} \tag{37}$$

where $L_o := P_o \cdot G$. Suppressing the arguments α, $i\omega$, temporarily, Eq. (37) can be rewritten as

$$S = \frac{P_o/P}{L_o + (P_o/P)} \tag{38}$$

The QFT performance and stability specifications now translate to

$$\max_{\alpha \in \Omega} |S| = \max_{\alpha \in \Omega} \left| \frac{P_o/P}{L_o + (P_o/P)} \right| \leq M(\omega), \ \forall \omega \geq 0 \tag{39}$$

Define the admissible loop transmission set as

$$B_p(\omega, \phi) = \left\{ L_o : \max_{\alpha \in \Omega} \left| \frac{P_o/P}{L_o + (P_o/P)} \right| \leq M, \forall \phi \epsilon [0, -2\pi], \forall \omega \epsilon [0 \quad \infty) \right\} \tag{40}$$

and $\partial B_p(\omega, \phi)$ as the boundary of $B_p(\omega, \phi)$. This boundary represents the set of all $\{L_o\}$ where Eq. (39) is satisfied with equality. Any optimum loop transmission function L_{opt} that solves the QFT optimization problem (27-29) must lie on $\partial B_p(\omega, \phi) \, \forall \omega \epsilon [0 \quad \infty)$.[28,33] The generation of $\partial B_p(\omega, \phi)$ proceeds as follows. At any frequency $\omega = \omega_k$, and any phase angle $\phi_l \epsilon [0 \quad -2\pi]$, $k = 1, 2, \ldots, N$, $l = 1, 2, \ldots, J$, L_o is selected by a search technique such that

$$\max_{\alpha \in \Omega} |S(i\omega_k)| = M(\omega_k) \tag{41}$$

By repeating the search procedure at different ϕ_l for any given ω_k, a closed continuous curve $\partial B_p(\omega_k)$ can be generated in the complex plane. This curve demarcates the complex plane into two disjoint regions: an interior and an exterior region. Any admissible $|L_o|$ is required to lie (preferably) on the boundary $\partial B_p(\omega_k)$ or at least in the exterior region $B_p(\omega, \phi)$ in order for the corresponding sensitivity function to satisfy the QFT specifications at $\omega = \omega_k$. For feasibility, the interior region is a forbidden region. By repeating the same procedure at a finite number of frequencies, a series of nonintersecting performance boundaries are generated on the complex plane. These boundaries are translated easily to the Nichols chart. The relative stability specification $|S(i\omega_h)| \leq M_p$ also translates to a corresponding (closed) region $B_s(\omega)$ surrounding the critical point (0 dB, -180 deg) on the Nichols chart called the relative stability region with boundary $\partial B_s(\omega)$. Any admissible L_o is also required to avoid the interior of $B_s(\omega)$. The performance and stability boundary curves, together with a candidate loop transmission function on the Nichols chart, are shown in Fig. 2.

The process of selecting an appropriate L_o from the set of all admissible L that satisfies the boundary conditions with mini-

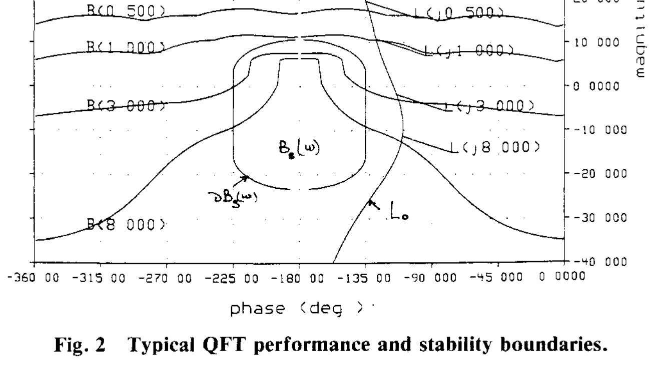

Fig. 2 Typical QFT performance and stability boundaries.

mum controller gain and minimum controller bandwidth is the very essence of QFT loop shaping. Traditionally, this aspect of the design involves trial and error procedures. In an attempt to systematize the process, we make the loop shaping procedure automatic by first solving an associated H^∞ optimization problem. The H^∞ optimal controller is then used as an initializing controller in a nonlinear optimization automatic loop shaping algorithm developed by Thompson.[29] The initializing H^∞ controller is needed in the routine because, unless an initial controller is near to the optimum in the low-frequency region, the optimization scheme may converge to some unacceptable local optimum. The H^∞ controller often satisfies the nearness condition and, hence, avoids the problem. To determine the H^∞ controller, we make use of the following standard result from H^∞ control theory. Let

$$\delta_G(s) := \max_{\alpha \in \Omega} \left| \frac{P(\alpha,s) - P_o(s)}{P_o(s)} \right|, \forall s \tag{42}$$

Theorem 1: Assume that all plants in the family

$$P = \left\{ P(\alpha,s): \max_{\alpha \in \Omega} \left| \frac{P(\alpha,s) - P_o(s)}{P_o(s)} \right| \leq \delta_G(s) \right\}, \forall s \tag{43}$$

have the same number of right half plane (RHP) poles. Let $W(s) \in RH^\infty$ satisfy $M^{-1}(\omega) \leq |W(s)|$, $\forall s$. Then the closed-loop system will meet the performance specification

$$\max_{\alpha \in \Omega} \| W(s) S(\alpha,s) \|_\infty < 1 \tag{44}$$

if and only if

$$H_o = \frac{P_o G_H}{1 + P_o G_H} \tag{45}$$

is stable and

$$\left\{ \delta_G(s) |H_o| + |M^{-1}(\omega) S_H(s)| \right\} < 1, \forall s \tag{46}$$

where

$$S_H = \frac{1}{1 + P_o G_H} \tag{47}$$

and G_H is the H^∞ controller.

Constructive necessary and sufficient conditions for the existence of G_H satisfying these theorem conditions are not known. However, by generating suitable RH^∞ weighting functions $W(s)$ and $V(s)$, the following H^∞ minimization problem can be set up:

$$G_H \in \min_G \left\{ \left[\|WS\|_\infty + \|VH\|_\infty \right] \right\} \tag{48}$$

where G is the set of all stabilizing controllers for P.

In standard H^∞ control, the choice of W and V is often qualitative. We can make it quantitative by considering the bounds given in the QFT specifications as follows.

Select $V(s) \in RH^\infty$ and satisfying

$$\delta_G(\omega) \leq |V(i\omega)| \; \forall \omega \geq 0 \tag{49}$$

Assume that $\delta_G(\omega) \in L_\infty$ and satisfies

$$\int_{-\infty}^{\infty} \log \delta_G(\omega) \, d\omega > -\infty \tag{50}$$

Then it is known that the function

$$\tilde{V}(s) := \exp\left[\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1 - i\lambda s}{s - i\lambda} \frac{\log \delta_G(\lambda)}{1 + \lambda^2} \, d\lambda \right] \tag{51}$$

as well as $\tilde{V}^{-1}(s)$ are both in H^∞,[20,27] such that

$$|\tilde{V}(i\omega)| = \delta_G(\omega) \tag{52}$$

almost everywhere. Next, approximate $\tilde{V}(s)$ uniformly by an RH^∞ function $V(s)$ such that

$$|\tilde{V}(i\omega)| \leq |V(i\omega)| \; \forall \omega \geq 0 \tag{53}$$

Now the sensitivity specifications

$$(27\text{-}31) => \|M^{-1}(\omega) S(\alpha, i\omega)\|_\infty \leq 1 \, \forall \alpha \in \Omega$$

Define $M^{-1}(\omega) := \hat{M}(\omega)$. Similarly, assume that $\hat{M}(\omega) \in L_\infty$, and satisfies

$$\int_{-\infty}^{\infty} \log \hat{M}(\omega) \, d\omega > -\infty \tag{54}$$

so that

$$\tilde{W}(s) := \exp\left[\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1 - i\lambda s}{s - i\lambda} \frac{\log \hat{M}(\lambda)}{1 + \lambda^2} \, d\lambda \right] \tag{55}$$

and $\tilde{W}^{-1}(s) \in H^\infty$. Proceed to cary out a uniform approximation to $\tilde{W}(s)$ by $W(s)$. Now insert $W(s)$ and $V(s)$ at the appropriate place in the H^∞ minimization problem (48), and use either the polynomial method of Kwakernaak[30] or the broadband matching method of Verma and Jonckheere[31] to obtain an appropriate controller G_H. Alternatively solve the simpler H^∞ problem[3]:

$$G \in \min_G \left\| \begin{matrix} WS \\ VH \end{matrix} \right\|_\infty = \mu \tag{56}$$

Indeed, $\mu < ½$ is a sufficient condition for the original H^∞ problem (48) to be solved.[3] Once G_H is determined, we draw the graph of $L_H(i\omega) = P_o G_H(i\omega)$ on the Nichols chart. This forms the initializing loop transmission function for the QFT optimization algorithm. On the same Nichols chart is superimposed the standard QFT performance and stability boundaries obtained from the templates from Eqs. (39) and (40). The overdesign in the H^∞ solution will usually be graphically apparent. The QFT optimization routine strives to reduce the gain-bandwidth area of L_H by moving $L_H(i\omega)$ toward the boundaries at every frequency. Once the boundary conditions are satisfied, the loop transmission function can be rolled off as rapidly as possible in order to reduce the controller bandwidth, as shown in Fig. 3. The resultant loop transmission function is designated as L_Q. It is not difficult to show that

$$|L_H(i\omega)| \geq |L_Q(i\omega)| \; \forall \omega \geq 0 \tag{57}$$

Starting with the same nominal models P_o, Eq. (57) then establishes that

$$|G_H(i\omega)| \geq |G_Q(i\omega)|, \forall \omega \geq 0 \tag{58}$$

Thus, although general existence conditions for solvability of the QFT optimization problem [(27) and (28)] are unknown, the existence of an H^∞ solution, as given in Theorem 1, is certainly a sufficient condition for solvability of the corresponding QFT problem. The reason for this lies in the different descriptions of the relative uncertainty P_o/P in Eq. (37). In QFT, P_o/P is some contour centered at $(1 + i0)$ in the complex plane. On the other hand, the corresponding H^∞ description of P_o/P is a disk centered at $(1 + i0)$ whose radius is given by the maximum distance from $(1 + i0)$ to the boundary of P_o/P. Consequently, the H^∞ uncertainty template at any frequency entirely contains the corresponding QFT uncertainty template at that frequency. As the loop transmission gain required to satisfy constraints (26) at any frequency is directly propor-

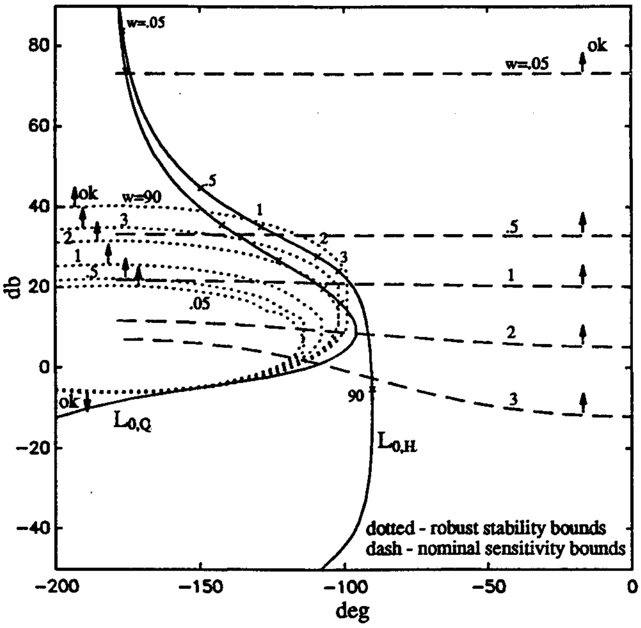

Fig. 3 L_H and L_Q superimposed on the problem boundaries.

tional to the spread or amount of uncertainty in P_o/P at that frequency, inequality (57) can be established easily.

V. Design Example

Here we present an example that details how uncertainty representation and the difference in the subsequent design philosophies can produce vastly different costs of feedback. The example is a standard problem for control system design.[32] Both the H^∞ design techniques and the QFT method were applied to this example. Both methods were able to meet the design specifications, with QFT doing slightly better in meeting the performance specifications. However, the difference in the cost of feedback between the QFT and the H^∞ solutions is quite large, with QFT producing a significantly smaller cost of feedback than the corresponding H^∞ solution. This is shown dramatically by comparing the respective controller gains and bandwidths for the H^∞ solution and QFT solution as done in Fig. 4. The H^∞ problem was solved first and then used as the initial design for the QFT optimization. After three iterations, the controller shown in Fig. 4 was obtained.

Consider a plant transfer function:

$$P(s,k,a) = \frac{ka}{s(s+a)}, \quad k \in [1,10] \quad a \in [1,10] \tag{59}$$

Here, $\alpha = [k,a]^T \in \Omega \subset IR^2$. It is desired to find a suitable controller $G(s)$ such that

$$\max_{\alpha \in \Omega} |S(i\omega,\alpha)| \le 0.089|i\omega|^2 \quad \forall \omega \ge 0 \tag{60}$$

$$\max_{\alpha \in \Omega} |H(i\omega,\alpha)| \le \frac{2.5}{|i\omega|^2}, \quad \forall \omega \ge 0 \tag{61}$$

By choosing the nominal model as

$$P_o(s) = \frac{10}{s(0.1s+1)} \tag{62}$$

i.e., with k and a at their maximum values, both QFT and H_∞ were able to solve the problem. For this model, the unstructured H^∞ perturbation is

$$\Delta_m(s) = \frac{0.9\left[(i\omega/0.91)+1\right]}{\left[(i\omega/1.0)+1\right]} \tag{63}$$

Whereas for QFT, any member of $P(s,k,a)$ is an admissible nominal model, for H^∞, one must use the maximum plant as given by $P_o(s)$, or a specific model for which the H^∞ weighting functions yield an admissible solution for otherwise $|\Delta_m(0)| > 1$. But for robust stabilty,

$$|H(i\omega)| < \frac{1}{|\Delta_m(i\omega)|}$$

which would then imply that $|H(0)| < 1$, thus contradicting the performance requirement following from $S(i\omega)+H(i\omega) \equiv 1$, that $H(0)=1$. For details, see Ref. 32. Using the same nominal models, the H^∞ solution with the weights:

$$\left| \frac{W(\omega)S(i\omega)}{V(\omega)H(i\omega)} \right|$$

where

$$W(\omega) = \frac{1}{0.089|(i\omega)|^2}$$

$$V(\omega) = \frac{1}{|\Delta_m(i\omega)|}$$

is given by

$$G_H = 22123.81\left[\frac{(s+3E-7)(s+1)(s+1.54)(s+10)}{(s+8E-10)(s+1E-7)(s+2)(s+4498)}\right] \tag{64}$$

Using this as the initial controller produced the following QFT design after a three iterate visual optimization:

$$G_Q = 1.2\left\{\frac{\left[(s/0.65)+1\right]\left[(s/8)+1\right]}{s\left[(s^2/60^2)+(s/60)+1\right]}\right\} \tag{65}$$

In this rather simple problem, the full power of the loop shaping optimization algorithm turned out not to be needed. For the situation when the algorithm is imperative, see Ref. 33. Examination of Figs. 3 and 5 clearly show that

$$\int_0^{\omega c} \log|S_H^{-1}|\, d\omega \ge \int_0^{\omega c} \log|S_Q^{-1}|\, d\omega \tag{66}$$

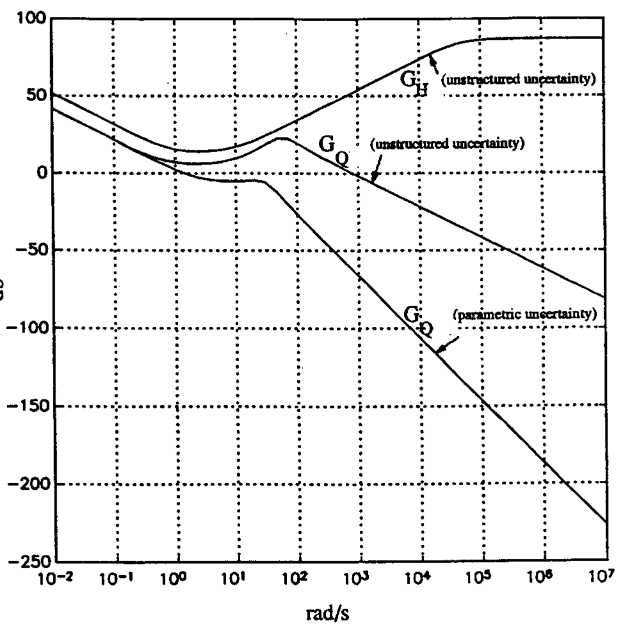

Fig. 4 Frequency response of G_H and G_Q compared with G_Q from tradition QFT.

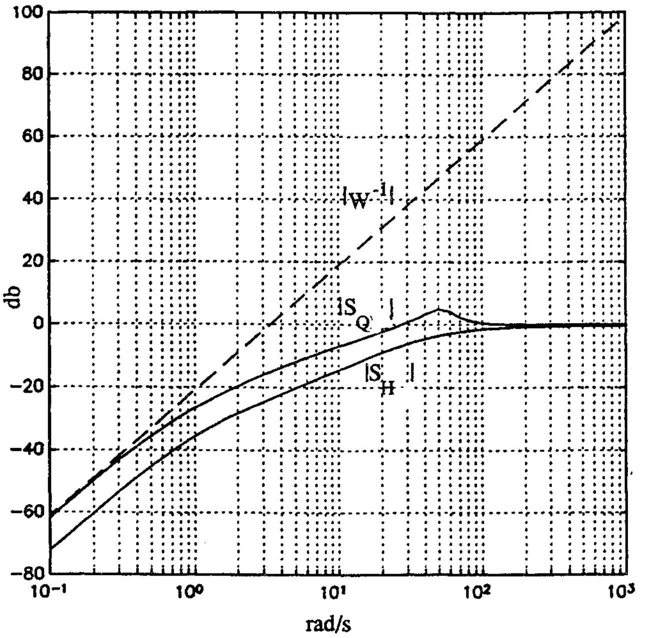

Fig. 5 Comparison of the sensitivities S_H and S_Q.

This implies that the QFT cost of feedback is much less than the corresponding H^∞ cost of feedback when both address the same nominal models and the same performance specifications as expected, where S_H and S_Q are, respectively, the H^∞ and QFT sensitivity functions. Observe that G_Q is strictly proper as desired, whereas G_H is not (it is only proper). This procedure can be extended to the multivariable case.[34]

VI. Conclusions

Quantitative feedback theory is a very useful robust design methodology whenever large parametric uncertainty and hard constraints on closed-loop response are indicated. Until recently, however, the technique relied almost entirely on semianalytical and graphical methods, making comparison with H^∞ control indirect, at best. By using H^∞ control as an initial trial design, however, the QFT methodology can be systematized. The formalization of the QFT process such as is presented here makes comparison with H^∞ and μ synthesis straightforward. The result of the noninclusion of phase information in these methods is the inevitability of a higher cost of feedback, as shown by the example. The extension of these ideas to multivariable systems is not difficult. Investigations into the adaptation of this approach to nonlinear control design along the lines originally suggested by Horowitz is in progress.

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